Graph algebras, groupoids, and symbolic dynamics

Toke Meier Carlsen

Western Sydney Mathematics Talks

Online Zoom seminar

14 May 2020

UNIVERSITY OF
THE FAROE ISLANDS

Introduction

In this talk, I will give an overview of some recent results that link diagonal-preserving isomorphism of graph algebras and isomorphism and equivalence of graph groupoids with continuous orbit equivalence, (eventual) conjugacy, and flow equivalence of symbolic dynamical systems of directed graphs.

The talk is primarily based on the following papers.

1. K. Matsumoto: *Orbit equivalence of topological Markov shifts and Cuntz-Krieger algebras* (2010).
2. K. Matsumoto and H. Matui: *Continuous orbit equivalence of topological Markov shifts and Cuntz-Krieger algebras* (2014).
3. N. Brownlowe, T. Carlsen, and M. Whittaker: *Graph algebras and orbit equivalence* (2017).
4. T. Carlsen and J. Rout: *Diagonal-preserving gauge-invariant isomorphisms of graph C^*-algebras* (2017).

Introduction

5. T. Carlsen and J. Rout: *Diagonal-preserving graded isomorphisms of Steinberg algebras* (2018).
6. T. Carlsen S. Eilers, E. Ortega, and G. Restorff: *Flow equivalence and orbit equivalence for shifts of finite type and isomorphism of their groupoids* (2019).

I will present a more extensive list of references at the end of the talk.

Graph algebras

By a *graph* we mean a quadruple (E^0, E^1, r, s) where E^0 and E^1 are sets, and r and s are maps from E^1 to E^0.
The Leavitt path algebra of a graph E with coefficients in a unital commutative ring R is the universal R-algebra $L_R(E)$ generated by a family $\{p_v : v \in E^0\}$ of pairwise orthogonal idempotents and a family $\{s_e, s_e^* : e \in E^1\}$ of elements satisfying

1. $p_{s(e)} s_e = s_e p_{r(e)} = e$ for $e \in E^1$,
2. $p_{r(e)} s_e^* = s_e^* p_{s(e)} = s_e^*$ for $e \in E^1$,
3. $s_e^* s_f = \delta_{e,f} p_{r(e)}$ for $e, f \in E^1$,
4. $p_v = \sum_{s^{-1}(v)} s_e s_e^*$ for $v \in E^0_{\text{reg}} := \{v \in E^0 : s^{-1}(v) \text{ is finite and non-empty}\}$.

The map $p_v \mapsto p_v$, $s_e \mapsto s_e^*$ extends to an involution on $L_R(E)$.

Graph algebras

The C^*-algebra of E is the universal C^*-algebra $C^*(E)$ generated by a family $\{p_v : v \in E^0\}$ of mutually orthogonal projections and a family $\{s_e : e \in E^1\}$ of partial isometries with mutually orthogonal ranges satisfying

1. $s_e^* s_e = p_{r(e)}$ for $e \in E^1$,
2. $s_e s_e^* \leq p_{s(e)}$ for $e \in E^1$,
3. $p_v = \sum_{e \in s^{-1}(v)} s_e s_e^*$ for $v \in E^0_{\mathrm{reg}}$.

Diagonal subalgebras and diagonal-preserving isomorphisms

- We denote by E^* the set of finite paths in E, and for $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \in E^*$ we let $s_\mu := s_{\mu_1} \cdots s_{\mu_n}$.
- The subalgebra $D_R(E) := \operatorname{span}_R\{s_\mu s_\mu^* : \mu \in E^*\} \subseteq L_R(E)$ is commutative.
- The C^*-subalgebra $\mathcal{D}(E) := \overline{\operatorname{span}}\{s_\mu s_\mu^* : \mu \in E^*\} \subseteq C^*(E)$ is commutative.
- An isomorphism $\phi : L_R(E) \to L_R(F)$ is *diagonal-preserving* if $\phi(D_R(E)) = D_R(F)$, and an isomorphism $\phi : C^*(E) \to C^*(F)$ is *diagonal-preserving* if $\phi(\mathcal{D}(E)) = \mathcal{D}(F)$.

The boundary path space of a graph

- An *infinite path* in E is an infinite sequence $x_1x_2\cdots$ of edges in E such that $r(e_i) = s(e_{i+1})$ for all i. We let E^∞ be the set of all infinite paths in E. The source map extends to E^∞ in the obvious way.
- The *boundary path space* of E is the space

$$\partial E := E^\infty \cup \{\mu \in E^* : r(\mu) \notin E^0_{\text{reg}}\}.$$

- If $\mu = \mu_1\mu_2\cdots\mu_m \in E^*$, $x = x_1x_2\cdots \in E^* \cup E^\infty$ and $r(\mu) = s(x)$, then we let μx denote the path $\mu_1\mu_2\cdots\mu_m x_1x_2\cdots \in E^* \cup E^\infty$.
- For $\mu \in E^*$, the *cylinder set* of μ is the set

$$Z(\mu) := \{\mu x \in \partial E : x \in r(\mu)\partial E\},$$

where $r(\mu)\partial E := \{x \in \partial E : r(\mu) = s(x)\}$.

The boundary path space of a graph

- For $\mu \in E^*$, the *cylinder set* of μ is the set

$$Z(\mu) := \{\mu x \in \partial E : x \in r(\mu)\partial E\},$$

  where $r(\mu)\partial E := \{x \in \partial E : r(\mu) = s(x)\}$.
- Given $\mu \in E^*$ and a finite subset $F \subseteq r(\mu)E^1 := \{e \in E^1 : s(e) = r(\mu)\}$ we define

$$Z(\mu \setminus F) := Z(\mu) \setminus \left(\bigcup_{e \in F} Z(\mu e) \right).$$

- ∂E is a locally compact Hausdorff space with the topology given by the basis $\{Z(\mu \setminus F) : \mu \in E^*, \ F \text{ is a finite subset of } r(\mu)E^1\}$, and each such $Z(\mu \setminus F)$ is compact and open.
- $D_R(E)$ is isomorphic to $LC_C(\partial E, R)$ and $\mathcal{D}(E)$ is isomorphic to $C_0(\partial E)$ by isomorphisms that map $s_\mu s_\mu^*$ to the characteristic function of $Z(\mu)$.

The shift maps

- For $n \in \mathbb{N}$, let $\partial E^{\geq n} := \{x \in \partial E : |x| \geq n\}$.
- Then $\partial E^{\geq n} = \cup_{\mu \in E^n} Z(\mu)$ is an open subset of ∂E.
- For $n \geq 1$, we define the *n-shift map* on E to be the map $\sigma_n : \partial E^{\geq n} \to \partial E$ given by $\sigma_n(x_1 x_2 x_3 \cdots x_n x_{n+1} x_{n+2} \cdots) = x_{n+1} x_{n+2} \cdots$ for $x_1 x_2 x_3 \cdots x_n x_{n+1} x_{n+2} \cdots \in \partial E^{\geq n+1}$ and $\sigma_n(\mu) = r(\mu)$ for $\mu \in \partial E \cap E^n$.
- We let σ_0 denote the identity map on ∂E.
- Then $\sigma_n : \partial E^{\geq n} \to \partial E$ is a local homeomorphism for all $n \in \mathbb{N}$.

The groupoid of a graph

The groupoid of E is the amenable locally compact Hausdorff étale groupoid

$$G(E) = \{(x,k,y) : \text{there exist } m,n \in \mathbb{N} \text{ such that } x \in \partial E^{\geq m}, \\ y \in \partial E^{\geq n},\ k = m-n,\ \sigma_m(x) = \sigma_n(y)\}$$

with $G(E)^{(0)} = \partial E$, $r(x,k,y) = x$, $s(x,k,y) = y$, $(x,k,y)^{-1} = (y,-k,x)$, and $(x,k,y)(y,l,z) = (x,k+l,z)$, and a basis consisting of compact open sets of the form

$$Z(U,m,n,V) = \{(x,m-n,y) : x \in U,\ y \in V, \sigma_m(x) = \sigma_n(y)\},$$

where $m,n \in \mathbb{N}$, U is a compact open subset of $\partial E^{\geq m}$ such that $(\sigma_m)|_U$ is injective, V is a compact open subset of $\partial E^{\geq n}$ such that $(\sigma_n)|_V$ is injective, and $\sigma_m(U) = \sigma_n(V)$.

The groupoid of a graph

- The Steinberg algebra $A_R(G(E))$ of $G(E)$ is isomorphic to $L_R(E)$ by an isomorphism that maps $LC_C(G(E)^{(0)}, R)$ onto $D_R(E)$, and
- the C^*-algebra $C^*(G(E))$ of $G(E)$ is isomorphic to $C^*(E)$ by an isomorphism that maps $C_0(G(E)^{(0)})$ onto $\mathcal{D}(E)$.

Continuous orbit equivalence

A *continuous orbit equivalence* is a homeomorphism $h : \partial E \to \partial F$ such that there are locally constant maps $k, l : \partial E^{\geq 1} \to \mathbb{N}$ and $k', l' : \partial F^{\geq 1} \to \mathbb{N}$ such that

$$\sigma_{l(x)}(h(x)) = \sigma_{k(x)}(h(\sigma_1(x)))$$

for $x \in \partial E^{\geq 1}$, and

$$\sigma_{l'(x')}(h^{-1}(x')) = \sigma_{k'(x')}(h^{-1}(\sigma_1(x')))$$

for $x' \in \partial F^{\geq 1}$.

- $x \in \partial E$ is *eventually periodic* if there are $m \neq n$ such that $\sigma_m(x) = \sigma_n(x)$.
- A continuous orbit equivalence $h : \partial E \to \partial F$ is said to *preserve isolated eventually periodic points* if h and h^{-1} map isolated eventually periodic points to isolated eventually periodic points.

Diagonal-preserving isomorphism, groupoid isomorphism, and continuous orbit equivalence

A unital ring R is indecomposable if 0 and 1 are the only idempotents in R.

Theorem [Matsumoto, Matsumoto&Matui, Brownlowe et al., Carlsen&Winger, Arnklint et al., Steinberg]

Let E and F be two graphs and R an indecomposable unital ring. TFAE:

1. There is a continuous orbit equivalence $h : \partial E \to \partial F$ that preserves isolated eventually periodic points.
2. $G(E)$ and $G(F)$ are topologically isomorphic.
3. There is a diagonal preserving isomorphism from $C^*(E)$ to $C^*(F)$.
4. There is a diagonal preserving R-algebra $*$-isomorphism from $L_R(E)$ to $L_R(F)$.
5. There is a diagonal preserving ring-isomorphism from $L_R(E)$ to $L_R(F)$.

Gradings, gauge-actions, cocycles, and eventually conjugacy

- There is a $\mathbb{Z}$-grading $\bigoplus_{n\in\mathbb{Z}} L_R(E)^{(n)}$ of $L_R(E)$ given by $L_R(E)^{(n)} = \mathrm{span}_R\{s_\mu s_\nu^* : \mu,\nu \in E^*,\ |\mu| - |\nu| = n\}$.
- There is an action $\gamma : \mathbb{T} \to \mathrm{Aut}(C^*(E))$ called the *gauge-action* satisfying $\gamma_z(p_v) = p_v$ and $\gamma_z(s_e) = zs_e$ for $z \in \mathbb{T}$, $v \in E^0$, and $e \in E^1$.
- The map $c_E : (x, n, y) \mapsto n$ is a *cocycle* from $G(E)$ to $\mathbb{Z}$.
- An *eventual conjugacy* is a homeomorphism $h : \partial E \to \partial F$ such that there are locally constant maps $k : \partial E^{\geq 1} \to \mathbb{N}$ and $k' : \partial F^{\geq 1} \to \mathbb{N}$ such that

$$\sigma_{k(x)+1}(h(x)) = \sigma_{k(x)}(h(\sigma_1(x)))$$

for $x \in \partial E^{\geq 1}$, and

$$\sigma_{k'(x')+1}(h^{-1}(x')) = \sigma_{k'(x')}(h^{-1}(\sigma_1(x')))$$

for $x' \in \partial F^{\geq 1}$.

Gradings, gauge-actions, cocycles, and eventually conjugacy

Theorem [Matsumoto, Carlsen&Rout, Steinberg]

Let E and F be two graphs and R an indecomposable unital ring. TFAE:

1. E and F are eventually conjugate.
2. There is a topological isomorphism $\phi : G(E) \to G(F)$ such that $c_F \circ \phi = c_E$.
3. There is a gauge-invariant diagonal preserving isomorphism from $C^*(E)$ to $C^*(F)$.
4. There is a graded diagonal preserving R-algebra $*$-isomorphism from $L_R(E)$ to $L_R(F)$.
5. There is a graded diagonal preserving ring-isomorphism from $L_R(E)$ to $L_R(F)$.

Stabilisations of graphs

- We let $M_\infty(R)$ denote the algebra of finitely supported, countable infinite square matrices over R, and $D_\infty(R)$ the abelian subalgebra consisting of diagonal matrices.
- We let $\mathcal{K}$ denote the C^*-algebra of compact operators on $l^2(\mathbb{N})$, and $\mathcal{C}$ the maximal abelian subalgebra consisting of diagonal operators.
- We let $\mathcal{R}$ denote the discrete groupoid $\mathbb{N} \times \mathbb{N}$ with $\mathcal{R}^{(0)} = \mathbb{N}$, $r(m,n) = m$, $s(m,n) = n$, $(m,n)^{-1} = (n,m)$, and $(m,n)(n,o) = (m,o)$.
- Then $A_R(\mathcal{R})$ is isomorphic to $M_\infty(R)$ by an isomorphism that maps $LC_C(\mathcal{R}^{(0)}, R)$ onto $D_\infty(R)$, and $C^*(\mathcal{R})$ is isomorphic to $\mathcal{K}$ by an isomorphism that maps $C_0(\mathcal{R}^{(0)})$ onto $\mathcal{C}$.
- If E is a graph, then we let SE denote the graph with $SE^0 = \mathbb{N} \times E^0$, $E^1 = \{(n,v) : n \in \mathbb{N},\ v \in E^0\} \cup \{(0,e) : e \in E^1\}$, $r(n,v) = (n,v)$, $r(0,e) = (0,r(e))$, $s(n,v) = (n+1,v)$, and $s(0,e) = (0,s(e))$.
- Then $G(SE)$ is isomorphic to $G(E) \times \mathcal{R}$.

Diagonal-preserving stabil isomorphism, groupoid isomorphism, and continuous orbit equivalence

Theorem [Carlsen&Rout, Steinberg]

Let E and F be two graphs and R an indecomposable unital ring. TFAE:

1. There is a continuous orbit equivalence $h : \partial(SE) \to \partial(SF)$ that preserves isolated eventually periodic points.
2. $G(E) \times \mathcal{R}$ and $G(F) \times \mathcal{R}$ are topologically isomorphic.
3. There is a diagonal preserving isomorphism from $C^*(E) \otimes \mathcal{K}$ to $C^*(F) \otimes \mathcal{K}$.
4. There is a diagonal preserving R-algebra $*$-isomorphism from $L_R(E) \otimes M_\infty(R)$ to $L_R(F) \otimes M_\infty(R)$.
5. There is a diagonal preserving ring-isomorphism from $L_R(E) \otimes M_\infty(R)$ to $L_R(F) \otimes M_\infty(R)$.

Gradings, gauge-actions, cocycles, and eventually conjugacy

Theorem [Carlsen&Rout, Steinberg]

Let E and F be two graphs and R an indecomposable unital ring. TFAE:

1. SE and SF are eventually conjugate.
2. There is a topological isomorphism $\phi : G(E) \times \mathcal{R} \to G(F) \times \mathcal{R}$ such that $(c_F \times c_{\mathcal{R}}) \circ \phi = c_E \times c_{\mathcal{R}}$ where $(c_E \times c_{\mathcal{R}})(\eta, (m, n)) = c_E(\eta) + m - n$.
3. There is a diagonal preserving isomorphism $\phi : C^*(E) \otimes \mathcal{K} \to C^*(F) \otimes \mathcal{K}$ such that $(\gamma_z \otimes \tau_z) \circ \phi = \phi \circ (\gamma_z \otimes \tau_z)$ for $z \in \mathbb{T}$ where $\tau_z(\theta_{n+1,n}) = z\theta_{n+1,n}$.
4. There is a graded diagonal preserving R-algebra $*$-isomorphism from $L_R(E) \otimes M_\infty(R)$ to $L_R(F \otimes M_\infty(R))$ with respect to the grading $\bigoplus_{n\in\mathbb{Z}}(L_R(E)^{(n)} \otimes \mathrm{span}_R\{\delta_{m+n,m} : m \in \mathbb{N}\})$.
5. There is a graded diagonal preserving ring-isomorphism from $L_R(E) \otimes M_\infty(R)$ to $L_R(F) \otimes M_\infty(R)$ with respect to the grading $\bigoplus_{n\in\mathbb{Z}}(L_R(E)^{(n)} \otimes \mathrm{span}_R\{\delta_{m+n,m} : m \in \mathbb{N}\})$.

Gradings, gauge-actions, and cocycles

Theorem [Carlsen&Rout, Steinberg]

Let E and F be two graphs and R an indecomposable unital ring. TFAE:

1. There is a topological isomorphism $\phi : G(E) \times \mathcal{R} \to G(F) \times \mathcal{R}$ such that $(c_F \times 0) \circ \phi = c_E \times 0$ where $(c_E \times 0)(\eta, (m, n)) = c_E(\eta)$.
2. There is a diagonal preserving isomorphism $\phi : C^*(E) \otimes \mathcal{K} \to C^*(F) \otimes \mathcal{K}$ such that $(\gamma_z \otimes \mathrm{id}) \circ \phi = \phi \circ (\gamma_z \otimes \mathrm{id})$ for $z \in \mathbb{T}$.
3. There is a graded diagonal preserving R-algebra $*$-isomorphism from $L_R(E) \otimes M_\infty(R)$ to $L_R(F) \otimes M_\infty(R)$ with respect to the grading $\bigoplus_{n\in\mathbb{Z}}(L_R(E)^{(n)} \otimes M_\infty(R))$.
4. There is a graded diagonal preserving ring-isomorphism from $L_R(E) \otimes M_\infty(R)$ to $L_R(F) \otimes M_\infty(R)$ with respect to the grading $\bigoplus_{n\in\mathbb{Z}}(L_R(E)^{(n)} \otimes M_\infty(R))$.

The shift space of a finite graph

- Suppose $E = (E^0, E^1, r, s)$ is a finite graph (i.e., E^0 and E^1 are finite) with no sinks (i.e., s is surjective) and no sources (i.e., r is surjective).
- We let $E^\infty_{-\infty} := \{(e_n)_{n\in\mathbb{Z}} : e_n \in E^1,\ r(e_n) = s(e_{n+1})\}$ and define $\overline{\sigma} : E^\infty_{-\infty} \to E^\infty_{-\infty}$ by $\sigma((e_n)_{n\in\mathbb{Z}})_m = e_{m+1}$.
- Equip $E^\infty_{-\infty}$ with the topology generated by $\{\overline{Z}_m(\mu) : m \in \mathbb{Z},\ \mu \in E^*\}$ where $\overline{Z}_m(\mu) = \{(e_n)_{n\in\mathbb{Z}} \in E^\infty_{-\infty} : e_m e_{m+1} \cdots e_{m+|\mu|-1} = \mu\}$.
- Then $E^\infty_{-\infty}$ is a compact totally disconnected Hausdorff space and $\overline{\sigma}$ is a homeomorphism.

Conjugacy and flow equivalence

- We say that $E_{-\infty}^{\infty}$ and $F_{-\infty}^{\infty}$ are *conjugate* is there is a homeomorphism $h : E_{-\infty}^{\infty} \to F_{-\infty}^{\infty}$ such that $\overline{\sigma} \circ h = h \circ \overline{\sigma}$.
- We say that $E_{-\infty}^{\infty}$ and $F_{-\infty}^{\infty}$ are *flow equivalent* if there is a homeomorphism $h : (E_{-\infty}^{\infty} \times \mathbb{R})/\sim \to (F_{-\infty}^{\infty} \times \mathbb{R})/\sim$ that maps flow lines onto flow lines in an orientation preserving way, where $\sim$ is the equivalence relation on $E_{-\infty}^{\infty} \times \mathbb{R}$ generated by $(\overline{\sigma}(x), t) \sim (x, t+1)$, and a flow line is a set of the form $\{[x,t] : t \in \mathbb{R}\}$.

Diagonal-preserving stabil isomorphism, groupoid isomorphism, and flow equivalence

Theorem [Cuntz&Krieger, Matsumoto&Matui, Carlsen et al, Steinberg]

Let E and F be two finite graphs with no sinks and no sources and R an indecomposable unital ring. TFAE:

1. $E_{-\infty}^{\infty}$ and $F_{-\infty}^{\infty}$ are flow equivalent.
2. SE and SF are continuously orbit equivalent.
3. $G(E) \times \mathcal{R}$ and $G(F) \times \mathcal{R}$ are topologically isomorphic.
4. There is a diagonal preserving isomorphism from $C^*(E) \otimes \mathcal{K} \to C^*(F) \otimes \mathcal{K}$.
5. There is a diagonal preserving R-algebra $*$-isomorphism from $L_R(E) \otimes M_\infty(R)$ to $L_R(F) \otimes M_\infty(R)$.
6. There is a diagonal preserving ring-isomorphism from $L_R(E) \otimes M_\infty(R)$ to $L_R(F) \otimes M_\infty(R)$.

Diagonal-preserving stabil isomorphism, groupoid isomorphism, and conjugacy

Theorem [Cuntz&Krieger, Carlsen&Rout, Steinberg]

Let E and F be two finite graphs with no sinks and no sources and R an indecomposable unital ring. TFAE:

1. $E_{-\infty}^{\infty}$ and $F_{-\infty}^{\infty}$ are conjugate.
2. There is a topological isomorphism $\phi : G(E) \times \mathcal{R} \to G(F) \times \mathcal{R}$ such that $(c_F \times 0) \circ \phi = c_E \times 0$ where $(c_E \times 0)(\eta, (m, n)) = c_E(\eta)$.
3. There is a diagonal preserving isomorphism $\phi : C^*(E) \otimes \mathcal{K} \to C^*(F) \otimes \mathcal{K}$ such that $(\gamma_z \otimes \mathrm{id}) \circ \phi = \phi \circ (\gamma_z \otimes \mathrm{id})$ for $z \in \mathbb{T}$.
4. There is a graded diagonal preserving R-algebra $*$-isomorphism from $L_R(E) \otimes M_\infty(R)$ to $L_R(F) \otimes M_\infty(R)$ with respect to the grading $\bigoplus_{n\in\mathbb{Z}}(L_R(E)^{(n)} \otimes M_\infty(R))$.
5. There is a graded diagonal preserving ring-isomorphism from $L_R(E) \otimes M_\infty(R)$ to $L_R(F) \otimes M_\infty(R)$ with respect to the grading $\bigoplus_{n\in\mathbb{Z}}(L_R(E)^{(n)} \otimes M_\infty(R))$.

References

1. S. Arklint, S. Eilers, and E. Ruiz. *A dynamical characterization of diagonal-preserving *-isomorphisms of graph C^*-algebras*. Ergodic Theory Dynam. Systems **38** (2018), 2401–2421. doi:10.1017/etds.2016.141
2. N. Brownlowe, T. Carlsen, and M. Whittaker. *Graph algebras and orbit equivalence*. Ergodic Theory Dynam. Systems **37** (2017), 389–417. doi:10.1017/etds.2015.52
3. T. Carlsen S. Eilers, E. Ortega, and G. Restorff. *Flow equivalence and orbit equivalence for shifts of finite type and isomorphism of their groupoids*. J. Math. Anal. Appl. **469** (2019), 1088–1110. doi:10.1016/j.jmaa.2018.09.056
4. T. Carlsen and J. Rout. *Diagonal-preserving gauge-invariant isomorphisms of graph C^*-algebras*. J. Funct. Anal. **273** (2017), 2981–2993. doi:10.1016/j.jfa.2017.06.018

References

5. T. Carlsen and J. Rout. *Diagonal-preserving graded isomorphisms of Steinberg algebras*. Commun. Contemp. Math. **20** (2018), 1750064, 25. doi:10.1142/S021919971750064X
6. T. Carlsen and M. Winger. *Orbit equivalence of graphs and isomorphism of graph groupoids*. Math. Scand. **123** (2018), 239–248. doi:10.7146/math.scand.a-105087
7. J. Cuntz. *A class of C^*-algebras and topological Markov chains. II. Reducible chains and the Ext-functor for C^*-algebras*. Invent. Math. **63** (1981), 25–40. doi:10.1007/BF01389192
8. J. Cuntz and W. Krieger. *A class of C^*-algebras and topological Markov chains*. Invent. Math. **56** (1980), 251–268. doi:10.1007/BF01390048

References

9. K. Matsumoto. *Orbit equivalence of topological Markov shifts and Cuntz-Krieger algebras*. Pacific J. Math. **246** (2010), 199–225. doi:10.2140/pjm.2010.246.199
10. K. Matsumoto and H. Matui. *Continuous orbit equivalence of topological Markov shifts and Cuntz-Krieger algebras*. Kyoto J. Math. **54** (2014), 863–877. doi:10.1215/21562261-2801849
11. B. Steiberg. *Diagonal-preserving isomorphisms of etale groupoid algebras*. J. Algebra **518** (2019), 412–439. doi:10.1016/j.jalgebra.2018.10.024

Thank you for your attention.